(Proc. IEEE Conference on Decision and Control, San Diego, CA, Jan. 1979) LIDS-P-869

ON STABILITY THEORY*

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Abstract

It is found that under mild assumptions, feedback system stability can be concluded if one can 'topologically separate' the infinite-dimensional function space containing the system's dynamical input-output relations into two regions, one region containing the dynamical input-output relation of the 'feedforward' element of the system and the other region containing the dynamical output-input relation of the 'feedback' element. Nonlinear system stability criteria of both the input-output type and the state-space (Lyapunov) type are interpreted in this context. The abstract generality and conceptual simplicity afforded by the topological separation perspective clarifies some of the basic issues underlying stability theory and serves to suggest improvements in existing stability criteria. A generalization of Zames' conic-relation stability criterion is proved, laying the foundation for improved multivariable generalizations of the frequency-domain circle stability criterion for nonlinear systems.

1. Introduction

Examining the conditions of Zames' conic relation stability theorem [1] -- a powerful abstract result including among its corollaries the Popov, circle, passivity, and small-gain stability criteria (cf. [2]-[6]) -- we have been struck by the observation that the conditions of the conic relation theorem have an unexpectedly simple interpretation in terms of a topological separation of the space on which the systems input-output relations are defined. Our scrutiny of the classical Lyapunov stability theory (e.g., [7]-[8]) has revealed that a similar interpretation applies to the stability conditions imposed by the Lyapunov theory. Motivated by these discoveries, we have developed a unified theory of stability in which the Lyapunov functions and the contraction mappings of previous theories are replaced by 'separating' functionals. The abstract generality of our approach serves to clarify the roles in stability theory of such basic concepts as extended normed spaces, contraction mappings, and positive-definite, decrescent, radially unbounded Lyapunov functions.

The conceptually simple view of stability theory afforded by topological separations has made clear to us some generalizations Zames' results [1]-[2] -- for example, the 'sector stability criterion' described in this paper. Our sector stability criterion forms the basis for several powerful and useful multivariable generalizations of the circle stability criterion; these generalizations are described in [9] and are the subject of a forthcoming paper.

This paper is based primarily on chapters 1-4 of Part II of M.G. Safonov's Ph.D. dissertation [9].

2. Problem Formulation

Our results concern the stability of the following canonical two-subsystem multivariable feedback system (Figure 1a)

$$\left.\begin{array}{l}(\underline{y},\ \underline{x}) \in \overline{\underline{G}}\ (\underline{u}) \\ (\underline{x},\ \underline{y}) \in \overline{\underline{H}}\ (\underline{v})\end{array}\right\} \qquad (2.1)$$

where

$\underline{u}\in\mathcal{U}_e$ and $\underline{v}\in\mathcal{V}_e$ are disturbance inputs to the system;

$\underline{x}\in\mathcal{X}_e$ and $\underline{y}\in\mathcal{Y}_e$ are the outputs of the system;

$\overline{G}(\underline{u}) \subset \mathcal{Y}_e \times \mathcal{X}_e$ and $\overline{H}(\underline{v}) \subset \mathcal{X}_e \times \mathcal{Y}_e$ are nonlinear relations which are dependent of the disturbance inputs $\underline{u}\in\mathcal{U}_e$ and $\underline{v}\in\mathcal{V}_e$ respectively;

$\mathcal{U}_e$, $\mathcal{V}_e$, $\mathcal{X}_e$, and $\mathcal{Y}_e$ are vector spaces.[1]

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*Research supported in part by NASA/AMES grant NGL-22-009-124, by NASA Langley Research Center grant NSG-1312, and by NSF Grant ENG78-05628.

[1] A set $\overline{\underline{X}}$ is described as a vector space [10, p. 171] (or, equivalently, as a linear set [11, pp. 43-44]) if for any two of its elements $\underline{x}_1$ and $\underline{x}_2$, the sum $\underline{x}_1 + \underline{x}_2$ is defined and is an element of $\overline{\underline{X}}$, and similarly the product $a\underline{x}$ is defined, 'a' being a scalar; additionally, the following a axioms must hold:

(1) $(\underline{x}_1 + \underline{x}_2) + \underline{x}_3 = \underline{x}_1 + (\underline{x}_2 + \underline{x}_3)$ (associative addition);

(2) $\underline{x}_1 + \underline{x}_2 = \underline{x}_2 + \underline{x}_1$ (commutative addition);

(3) an element $\underline{0}$ exists in $\overline{X}$ such that $0\underline{x} = \underline{0}$ for all $\underline{x}$ in $\overline{X}$;

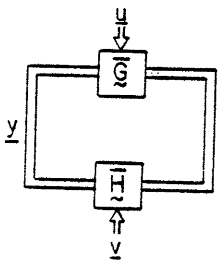
(4) $(a_1 + a_2)\underline{x} = a_1\underline{x} + a_2\underline{x}$ (distributive laws);

(5) $a(\underline{x}_1 + \underline{x}_2) = a\underline{x}_1 + a\underline{x}_2$

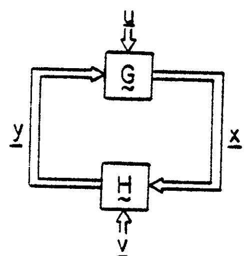
(6) $(ab)\underline{x} = a(b\underline{x})$ (associative multiplication);

(7) $1\underline{x} = \underline{x}$.

It is assumed that $(\underline{0}, \underline{0}) \in \overline{G}(\underline{0})$ and that $(\underline{0}, \underline{0}) \in \overline{H}(\underline{0})$ so that the pair $(\underline{x}, \underline{y}) = (\underline{0}, \underline{0})$ is an equilibrium solution of the undisturbed system. The system (2.1) defines a relation between input pairs $(\underline{u}, \underline{v}) \in U_e \times V_e$ and output pairs $(\underline{x}, \underline{y}) \in X_e \times Y_e$; equivalently (2.1) defines a subset of the space $(U_e \times V_e) \times (X_e \times Y_e)$.



(a) Canonical two-subsystem multivariable feedback system



(b) Special case: $\underset{\sim}{G}(\underline{y})$ and $\underset{\sim}{H}(\underline{x})$ operators

Figure 1

The vector spaces U_e, V_e, X_e, and Y_e are assumed to be extended normed spaces, defined in terms of collections of normed spaces U_τ, V_τ, X_τ, and Y_τ and a collection of linear 'projection' operators P_τ as follows.

Definition: Let Z_e be a vector space. If there is a associated with Z_e an interval T and a collection of linear operators P_τ $(\tau \in T)$ mapping Z_e into the collection of normed spaces Z_τ $(\tau \in T)$ then Z_e is the extended normed space induced by the collection of operators P_τ $(\tau \in T)$. If, additionally, each of the spaces Z_τ is an inner-product space then we say that Z_e is the extended inner product space induced by the collection of operators P_τ.[2] A vector space Z_e which is itself a normed space is presumed to be the extended normed space for which $Z_\tau = Z_e$ and P_τ is the identity operator, unless Z_τ and/or P_τ are specifically stated to be otherwise. For brevity, we denote $||\underline{z}||_\tau \triangleq ||P_\tau \underline{z}||_{Z_\tau}$ and $<\underline{z}_1, z_2>_\tau \triangleq <P_\tau z_1, P_\tau z_2>_{Z_\tau}$.

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[2]One can define a variety of 'extended norm' functionals $||\cdot||_e : Z_e \to R_+ \cup \{\infty\}$ on the space Z_e, e.g., [1]

$$||\underline{z}||_e = \sup_{\tau \in T} ||P_\tau \underline{z}||_\tau \quad ; \text{ or [5]}$$

$$||\underline{z}||_e = \limsup_{\tau \to (\sup T)} ||P_\tau \underline{z}||_\tau \quad .$$

Since $||\underline{z}||_e$ may in general be infinite, the functional $||\cdot||$ is not necessarily a norm in the usual sense. However, $||\cdot||_e$ does define a norm on the subspace

$$Z \triangleq \{\underline{z} \in Z_e \;\Big|\; ||\underline{z}||_e < \infty\}.$$

For purposes of stability analysis, we have found that it is not necessary to introduce the extended norm $||\cdot||_e$ or the normed subspace Z since the stability properties of each $\underline{z} \in Z_e$ can be determined from the τ-dependence of $||\underline{z}||_\tau$.

Comments: The foregoing problem formulation is considerably more general than is usual in stability theory. Typically, the interval T represents time and the spaces X_e and Y_e consist of functions mapping T into R^n; the operator P_τ is typically taken to be the linear truncation operator [1]

$$(P_\tau \cdot \underline{\xi})(t) = \begin{Bmatrix} \underline{\xi}\,(t), & \text{if } t \leq \tau \\ \underline{0}\;, & \text{if } t > \tau \end{Bmatrix} \qquad (2.2)$$

The interval T might typically be taken to be either the non-negative real numbers R_+ (in the case of continuous-time systems) or the set of non-negative integers Z_+ (in the case of discrete-time systems). The disturbance vector spaces U_e and V_e are typically both either R^n (in the case of Lyapunov state-space results [7]) or spaces of functions mapping T into R^n (in the case of 'input-output' stability results [6]).[3]

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[3]Readers unfamiliar with the notion of a relation, the concept of an extended normed space, the linear truncation operator, or other concepts and definitions associated with 'input-output' stability theory may find it helpful to refer to one of the books [3], [4], [6] or the concise and lucid original exposition of Zames [1].

In the special case in which the relations $\overline{\mathcal{G}}(\underline{u})$ and $\overline{\mathcal{H}}(\underline{v})$ are induced by disturbance-dependent functional operators $\mathcal{G}(\underline{u}): \mathcal{Y}_e \to X_e$ and $\mathcal{H}(\underline{v}): X_e \to \mathcal{Y}_e$, then the system (2.1) may be represented by the equivalent set of feedback equations

$$\left.\begin{aligned} \underline{x} &= \mathcal{G}(\underline{u}) \cdot \underline{y} \\ \underline{y} &= \mathcal{H}(\underline{v}) \cdot \underline{x} \end{aligned}\right\} \qquad (2.3)$$

(see Figure 1b).

3. Separation Interpretation of Stability Theory

To provide motivation and a conceptual framework for the development that follows, it is instructive to digress at this point by explaining how to give a simple geometric interpretation to the conic-relation stability theorem (Zames [1], Theorem 2a) and to one of the principal theorems of Lyapunov stability theory.[4]

[4] The circle Popov, passivity, and small-gain stability theorems as well as the sufficiency part of the Nyquist theorem follow as corollaries to Zames' conic relation stability theorem; this is demonstrated in [1]-[2]. These and still other results can be proved via the Lyapunov theory (cf. [12]).

Conic Relation Theorem

The conditions of the conic relation stability theorem involve 'conic' regions of the type (see Figure 2)

$$\text{Cone }(c,r) \triangleq \{(\underline{x},\underline{y}) \in X_e \times \mathcal{Y}_e \,\big|\, \|(\underline{y}-c\underline{x})\|_\tau \le r\|\underline{x}\|_\tau \text{ for all } \tau\in T\} = \{(\underline{x},\underline{y}) \in X_e \times \mathcal{Y}_e \,\big|\, \langle \underline{y}-(c+r)\underline{x},\ \underline{y}-(c-r)\underline{x}\rangle_\tau \le 0 \text{ for all } \tau\in T\} \qquad (3.1)$$

where c and r are scalars called the cone center and cone radius, respectively.[5] It is said that a relation $\overline{\mathcal{H}} \subset X_e \times \mathcal{Y}_e$ is inside Cone (c, r) if

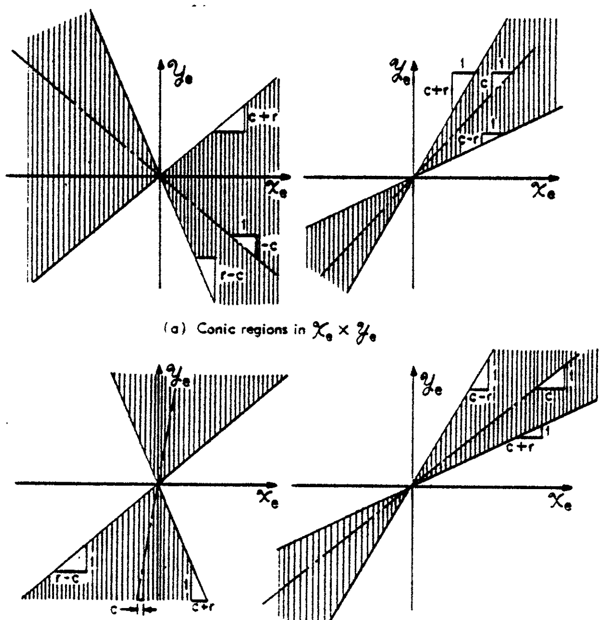
$$\overline{\mathcal{H}} \subset \text{Cone }(c, r); \qquad (3.2)$$

$\overline{\mathcal{H}}$ is strictly inside Cone (c, r) if for some $r' < r$,

$$\overline{\mathcal{H}} \subset \text{Cone }(c, r') \subset \text{Cone }(c, r) \qquad (3.3)$$

The notions of outside and strictly outside are defined analogously using in place of Cone (c, r) its complement. A property of conic regions such as (3.1) that is central to the geometric interpretation of Zames' conic-relation stability theorem is that the complement of such a conic region in $X_e \times \mathcal{Y}_e$ corresponds to either a conic region (if $c \le r$) or the complement of a conic region (if $c \ge r$) in $\mathcal{Y}_e \times X_e$, as may be seen by comparison of Figure 2a with Figure 2b.

[5] The notation Cone (c, r) is non-standard; Zames [1] uses the notation {c-r, c+r}.



(a) Conic regions in $X_e \times \mathcal{Y}_e$

(b) Conic regions in $\mathcal{Y}_e \times X_e$

Figure 2. Visualization of '$X_e \times \mathcal{Y}_e$ - plane' of conic regions with center c and radius r for several values of c and r.

For reasons that are not entirely apparent, the conditions of Zames' conic-relation theorem also require that $\mathcal{U}_e$, $\mathcal{V}_e$, X_e, and $\mathcal{Y}_e$ be identical extended inner product spaces and that the disturbances $\underline{u} \in X_e$ and $\underline{v} \in \mathcal{Y}_e$ enter additively; i.e., if $(\underline{x}_1, \underline{y}_1) \in \overline{\mathcal{H}}(\underline{v})$ and $(\underline{y}_2, \underline{x}_2) \in \overline{\mathcal{G}}(\underline{u})$, then, respectively,

$$(\underline{x}_1 + \underline{v},\ \underline{y}_1) \in \overline{\mathcal{H}}(\underline{0}) \qquad (3.4)$$

$$(\underline{y}_2 + \underline{u},\ \underline{x}_2) \in \overline{\mathcal{G}}(\underline{0}) \qquad (3.5)$$

(see Figure 3). Subject to these restrictions on the class of systems considered, the conditions of Zames' conic relation theorem state quite simply that a sufficient condition for the feedback system (2.1) to be stable is that (for appropriately chosen center and radius parameters c and r), the relation $\overline{\mathcal{H}}(\underline{0})$ be strictly inside Cone (c, r) $\subset X_e \times \mathcal{Y}_e$ and the relation $\overline{\mathcal{G}}(\underline{0})$ be inside the region of $\mathcal{Y}_e \times X_e$ corresponding to the complement of Cone (c, r) (see Figure 4). The interpretation of the conic relation stability theorem in terms of a topological separation is immediately evident: the interior of Cone (c, r) and the interior of the complement of Cone (c, r) form a topological separation [13] of the space $X_e \times \mathcal{Y}_e$ [less the equilibrium point $(\underline{x}, \underline{y}) =$

$(\underset{\sim}{0}, \underset{\sim}{0})$ and other points on the boundary of Cone (c, r)] into two disjoint regions, the closure of one region containing all non-zero pairs $(\underline{x}, \underline{y}) \in \bar{\underset{\sim}{H}}(\underset{\sim}{0})$ and the interior of the other region containing all non-zero pairs $(\underline{x}, \underline{y})$ such that $(\underline{y}, \underline{x}) \in \bar{\underset{\sim}{G}}(\underset{\sim}{0})$.

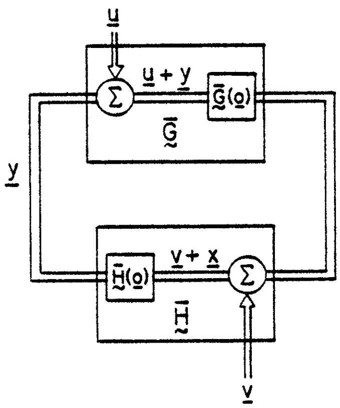


Figure 3. Feedback system with disturbances entering additively.

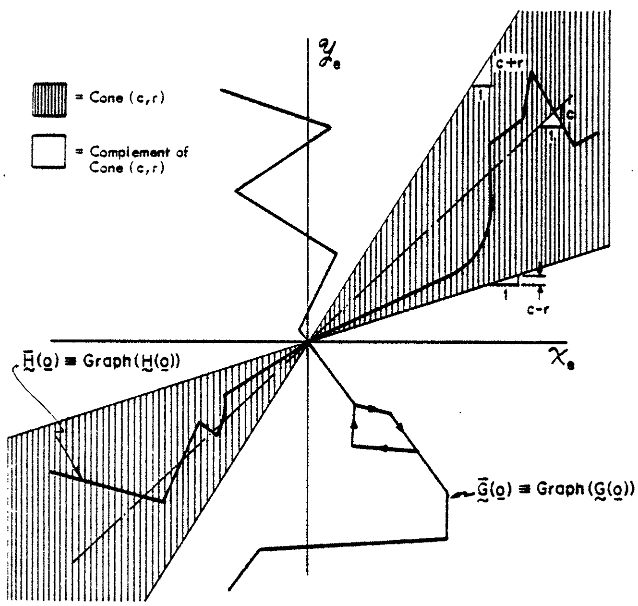


Figure 4. Two-dimensional geometric interpretation of Zames' conic relation theorem.

Lyapunov Stability Theorem

In the continuous-time state-space stability problems typically attacked by Lyapunov methods, the system under consideration is often given in the form (see Figure 5)

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), t); \qquad \underline{x}(0) = \underline{x}_0 \tag{3.6}$$

where

$$t \in R_+ \triangleq [0, \infty),$$
$$\underline{x}(t) \in R^n \text{ for all } t \in R_+ .$$

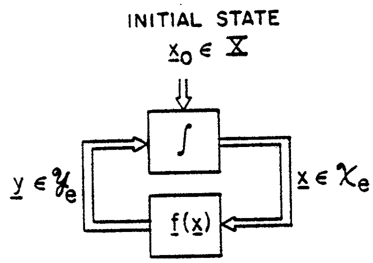


Figure 5. Feedback representation of systems considered by Lyapunov methods.

This system can be interpreted in terms of the system (2.3) as

$$\underline{x} \equiv [\underline{x}(t)]_{t\in R_+} = \left\{ \underline{x}_o + \int_o^t \underline{y}(\tau)d\tau \right\}_{t\in R_+} = \underset{\sim}{G}(\underline{u})\underline{y}$$

$$y \equiv [\underline{y}(t)]_{t\in R_+} = \left\{ \underline{f}[\underline{x}(t),t] \right\}_{t\in R_+} = \underset{\sim}{H}(\underline{v})\underline{x}$$

$$\underline{v} = \underline{0} \in \mathcal{V}_e = \{\underline{0}\}$$

$$\underline{u} = \underline{x}_o \in \mathcal{U}_e = R^n \tag{3.7}$$

where $T = R_+$, $\underline{x} \in \mathcal{X}_e$ and $\underline{y} \in \mathcal{Y}_e$ and where $\mathcal{X}_e$ and $\mathcal{Y}_e$ are the extended normed spaces

$$\mathcal{X}_e = \{\underline{x}: R_+ \to R^n \mid \underline{x} \text{ is once differentiable}\} \tag{3.8}$$

$$\mathcal{Y}_e = \{\underline{y}: R_+ \to R^n\} \tag{3.9}$$

induced by the identity operator

$$P_\tau \underline{x} = \underline{x} \text{ for all } \underline{x} \tag{3.10}$$

mapping $\mathcal{X}_e$ and $\mathcal{Y}_e$ into the normed spaces

$$\mathcal{X}_\tau = \mathcal{X}_e \tag{3.11}$$

with norm

$$||\underline{x}||_\tau = \sup_{t\in[0,\ \tau]} ||\underline{x}(t)||_{R^n} \tag{3.12}$$

and

$$\mathcal{Y}_\tau = \mathcal{Y}_e \tag{3.13}$$

with (degenerate) norm

$$||\underline{y}||_\tau \equiv 0 . \tag{3.14}$$

One of the main theorems of Lyapunov stability ([8], Theorem 4.1) states that if in some neighborhood of the origin of the state space R^n there exists a positive-definite decrescent Lyapunov function $V(\underline{x}, t)$ such that its derivative $\nabla V(\underline{x}, t)\underline{f}(\underline{x}, t) + \partial V/\partial t\ (\underline{x}, t)$ is negative semi-definite in this neighborhood, then the solution $\underline{x}(t) \equiv \underline{0}$ is stable in the sense of Lyapunov; i.e., for every constant $\varepsilon > 0$, there exists a constant $\delta(\varepsilon) > 0$ such that $||\underline{x}_0||_{R^n} \leq \delta(\varepsilon)$ implies $||\underline{x}||_\tau \leq \varepsilon$ for all τ.

The stipulation that $V(\underline{x}, t)$ be positive definite ensures that every pair $(\underline{x}, \underline{y})$ satisfying $\underline{x} = \underset{\sim}{G}(\underline{0})\cdot\underline{y}$ lies inside the set

$$\{(\underline{x}, \underline{y}) \in X_e \times Y_e \,\Big|\, \int_0^\tau [\nabla V(\underline{x}(t), t)]^T \underline{y}(t) + \partial V/\partial t(\underline{x}(t), t)dt \geq 0 \text{ for all } \tau \in T\} \quad (3.15)$$

since, for

$$\underline{x} = \underset{\sim}{G}(\underline{0})\underline{y}\ ,$$

$$\int_0^\tau \nabla V(\underline{x}(t), t)^T y(t) + \partial V/\partial t(\underline{x}(t), t)dt = V(x(\tau), \tau) - V(\underline{0}, 0) = V(x(\tau), \tau) \geq 0.$$

The derivative condition ensures that, in some neighborhood of the origin of $X_e \times Y_e$, every pair $(\underline{x}, \underline{y})$ satisfying $\underline{y} = \underset{\sim}{H}(\underline{0})\cdot\underline{x}$ lies 'outside' the set (3.15), i.e., is contained in the set

$$\{(\underline{x}, \underline{y}) \,\Big|\, \int_0^\tau \nabla V(\underline{x}(t), t)\underline{y}(t) + \partial V/\partial t(\underline{x}(t), t)dt \leq 0 \text{ for all } \tau \in T\}. \quad (3.16)$$

The results of this paper show that it is more than just a coincidence that the conditions of such powerful stability results as the conic-relation stability theorem and the foregoing Lyapunov stability theorem correspond to the existence of a topological separation. The results show in essence that one can use any such partitioning of $X_e \times Y_e$ into two disjoint regions, provided that the 'distance' between the two regions is positive and increases as the 'distance' from the equilibrium increases. What constitutes suitable measure of 'distance' is the subject of our main results in §5.

4. Notation and Terminology

In this section some of the standard terminology from the stability literature (e.g., [1], [7]) is reviewed and, where necessary, generalized so as to be applicable to the broad class of stability problems admitted by our abstract problem formulation.

Relations (cf. [1]): A relation $\overline{\underset{\sim}{R}}$ is any set of the form $\overline{\underset{\sim}{R}} \subset X \times Y$; i.e., a relation is any subset of the Cartesian product of any two sets. A relation $\overline{\underset{\sim}{R}} \subset X \times Y$ can be represented equivalently as a mapping of subsets of X into subsets of Y and in this regard is merely a generalization of the notion of a function mapping X into Y. Some operations involving relations are defined below.

Image: The image $\overline{\underset{\sim}{R}}[A]$ of a set $A \subset X$ under a relation $\overline{\underset{\sim}{R}} \subset X \times Y$ is the subset of Y

$$\overline{\underset{\sim}{R}}[A] \triangleq \{\underline{y} \,|\, (\underline{x}, \underline{y}) \in \overline{\underset{\sim}{R}} \text{ for some } \underline{x} \in A\}\ . \quad (4.1)$$

For $\underline{x}_0 \in X$, we may denote $\overline{\underset{\sim}{R}}[\{\underline{x}_0\}]$ using the abbreviated notation $\overline{\underset{\sim}{R}} \cdot \underline{x}_0$ or $\overline{\underset{\sim}{R}}\underline{x}_0$.

Inverse: The inverse of a relation $\overline{\underset{\sim}{R}} \subset X \times Y$ is the relation $\overline{\underset{\sim}{R}}^I \subset Y \times X$

$$\overline{\underset{\sim}{R}}^I \triangleq \{(\underline{y}, \underline{x}) \in Y \times X \,\big|\, (\underline{x}, \underline{y}) \in \overline{\underset{\sim}{R}}\}\ ; \quad (4.2)$$

clearly, the inverse always exists.

Composition Product: The composition product of the relations $\overline{\underset{\sim}{R}}_1\ X \times Y$ followed by $\overline{\underset{\sim}{R}}_2 \subset Y \times Z$ is the relation $\overline{\underset{\sim}{R}}_2 \circ \underset{\sim}{R}_1 \subset X \times Z$

$$\overline{\underset{\sim}{R}}_2 \circ \overline{\underset{\sim}{R}}_1 \triangleq \{(\underline{x}, \underline{z}) \in X \times Z \,|\, \text{there exists } \underline{y} \in Y \text{ such that } (\underline{x}, \underline{y}) \in \overline{\underset{\sim}{R}}_1 \text{ and } (y, z) \in \overline{\underset{\sim}{R}}_2\}\ . \quad (4.3)$$

Sum: If $\overline{\underset{\sim}{R}}_1, \overline{\underset{\sim}{R}}_2 \subset X \times Y$ and if addition is defined on Y, then the sum of the relations $\underset{\sim}{R}_1$ and $\underset{\sim}{R}_2$ is the relation

$$\overline{\underset{\sim}{R}}_1 + \overline{\underset{\sim}{R}}_2 \triangleq \{(\underline{x}, \underline{y}) \in X \times Y \,|\, \underline{x} \in X \text{ and } \underline{y} = \underline{y}_1 + \underline{y}_2 \text{ for some } \underline{y}_1 \in \overline{\underset{\sim}{R}}_1\ \underline{x} \text{ and } \underline{y}_2 \in \overline{\underset{\sim}{R}}_2\ \underline{x}\} \quad (4.4)$$

Graph: If $\underset{\sim}{G}$ is a function mapping of points $\underline{x} \in X$ into points $\underset{\sim}{G}\underline{x} \in Y$, then the graph of $\underset{\sim}{G}$ is the relation

$$\overline{\underset{\sim}{G}} \equiv \text{Graph}\ (\underset{\sim}{G}) \triangleq \{(\underline{x}, \underline{y}) \in X \times Y \,|\, \underline{x} \in X \text{ and } \underline{y} = \underset{\sim}{G}\underline{x}\}$$

Stability Terminology

Class K (cf. [7, p. 7]): A function ϕ mapping the non-negative real numbers R_+ into non-negative real numbers R_+ is defined to be in class K, denoted $\phi \in K$, if ϕ is continuous, strictly increasing and $\phi(0) = 0$.

Positive Definite; Decrescent; Radially Unbounded: Let X_e be an extended normed space and T be the associated interval; let S be a subset of X_e containing the point $\underline{x} = \underline{0}$; a functional η: $S \times T \to R$ is said to be positive definite on S if both

i) for some ϕ in class K, all $\underline{x} \in S$ and all $\tau \in T$

$$\eta(\underline{x}, \tau) \geq \phi(||\underline{x}||_\tau) \quad (4.6)$$

and

ii) for all $\tau \in T$, $\eta(\underline{0}, \tau) = 0$ (4.7)

a functional η: $X_e \times T \to R$ is said to be decrescent on S if for some ϕ in class K and all $\tau \in T$

$$0 \leq \eta(\underline{x}, \tau) \leq \phi(||\underline{x}||_\tau)\ . \quad (4.8)$$

A functional η: $X_e \times T \to R$ is said to be radially unbounded on S if there exists a continuous non-decreasing function ϕ: $R_+ \to R_+$ with $\lim_{\alpha \to \infty} \phi(\alpha) = \infty$ such that for all $\tau \in T$ and all $\underline{x} \in S$

$$\eta(\underline{x}, \tau) \geq \phi(||\underline{x}||_\tau)\ . \quad (4.9)$$

Neighborhood: For any extended normed space X_e, any set $A \subset X_e$, and any non-negative number α, the open neighborhood $N(A;\alpha)$ is the set

$$N(A;\alpha) \triangleq \{\underline{x} \in X_e \,|\, \text{for some } \underline{x}_0 \in A,\ ||(\underline{x} - \underline{x}_0)|| < \alpha \text{ for all } \tau \in T\}\ . \quad (4.10)$$

If A consists of a single point, say $\underline{x}_0$, we may use the abbreviated notation $N(\underline{x}_0;\alpha) \triangleq N(\{\underline{x}_0\};\alpha)$. A set S is said to be a neighborhood of a set A if for some $\varepsilon > 0$

$$N(A;\varepsilon) \subset S \; ; \tag{4.11}$$

if $\underline{x}_0$ is a point, the set S is said to be a neighborhood of $\underline{x}_0$ if, for some $\varepsilon > 0$, $N(\underline{x}_0;\ \varepsilon) \subset S$.

$||(\underline{x},\ \underline{y})||$: for any normed spaces X, Y and any $(\underline{x},\ \underline{y}) \in X \times Y$, the notation $||(\underline{x},\ \underline{y})||$ is in this paper defined to mean

$$||(\underline{x},\ \underline{y})|| \;\triangleq\; (||\underline{x}||^2 + ||\underline{y}||^2)^{1/2} \; ; \tag{4.12}$$

clearly, (2.10) defines a norm on $X \times Y$. So, for example, $X_e \times Y_e$ is an extended normed space with associated norm $||(\underline{x},\ \underline{y})||_\tau = (||\underline{x}||_\tau{}^2 + ||\underline{y}||_\tau{}^2)^{1/2}$.

Gain; Incremental Gain: Suppose X_e and Y_e are extended normed spaces; let $\overline{F} \subset X_e \times Y_e$. If for some scalar $k < \infty$ and for all $\underline{x} \in X_e$ and all $\tau \in T$,

$$\overline{F}\ \underline{x} \subset N(\{\underline{0}\}\ ;\ k||\underline{x}||_\tau)\ . \tag{4.13}$$

then $\overline{F}$ has finite gain; the infimal k for which (4.13) is satisfied is called the gain of $\overline{F}$, If for some $k < \infty$, all $\underline{x}_1$, $\underline{x}_2 \in X_e$, and all $\tau \in T$,

$$\overline{F}\ \underline{x}_2 \subset N(\overline{F}\ \underline{x}_1;\ k||\underline{x}_1 - \underline{x}_2||_\tau)\ , \tag{4.14}$$

then $\overline{F}$ has finite incremental gain; the infimal k satisfying (4.14) is called the incremental gain of $\overline{F}$. A function F: $X_e \to Y_e$ is said to have finite (incremental) gain if the relation Graph (F) has finite (incremental) gain.

Bounded; Stable; Finite-Gain Stable: Let X_e and Y_e be extended normed spaces; let $\overline{F} \subset X_e \times Y_e$; let $A \subset Y_e$. If there exist neighborhood of A, say S, and a non-decreasing continuous function ϕ: $R_+ \to R_+$ such that for all $\underline{x} \in S$

$$\overline{F}\ \underline{x} \subset N(A;\ \phi(||\underline{x}||_\tau)) \tag{4.15}$$

then $\overline{F}$ is bounded in S about the set A; if $\phi \in K$ then we say $\overline{F}$ is stable about the set A; if ϕ is linear (i.e., ϕ of the form $\phi(||\underline{x}||_\tau) = k||\underline{x}||_\tau$), then we say that $\overline{F}$ is finite-gain stable about the set A. If, in the foregoing, the neighborhood S can be taken to be the entire space X_e, then 'bounded in S' becomes simply bounded, 'stable' becomes globally stable and finite-gain stable becomes globally finite-gain stable. When $A = \{\underline{0}\}$, then we say simply that $\overline{F}$ is bounded in S, or stable, or finite-gain stable, respectively (i.e., we omit the phrase 'about the set A'). These definitions also apply to a function F when Graph (F) is substituted for $\overline{F}$ in the foregoing.

Comments: It is necessary that stability be defined here, because there is no standard definition in the literature. The motivation for the present choice of definition is two-fold. First, the definition is more flexible than previous definitions of stability in that i) inputs need not enter additively, and ii) by allowing discussion of stability about an arbitrary set, the definition permits one, in principle, to address certain special issues in stability theory, e.g., the stability about time-varying functions or sets of time-varying functions such as the limit cycles of autonomous systems. Second, the definition meshes well with the classical notion of stability in the sense of Lyapunov (cf. [8]), coinciding when the magnitude of the system input is taken to be the Euclidean norm of its initial state and the state trajectory is presumed to lie in the extended normed space X_e defined in (3.8) - (3.11). Indeed, it appears that by appropriate choice of extended normed spaces that nearly all the stability definitions employed in the literature (cf. [1]-[8], [12], [14]) can be made to coincide with boundedness, stability, or finite-gain stability as defined in this paper.

It is noteworthy that in the case of linear systems, the definitions of bounded, stable, and finite-gain stable coincide: for such systems ϕ can always be taken to be linear: e.g., in (4.15), pick any $\underline{x}_0$ with $||\underline{x}_0|| \neq 0$ and replace $\phi(||\underline{x}||_\tau)$ by $\phi'(||\underline{x}||_\tau) = [||\underline{x}||_\tau \div \phi(||\underline{x}_0||_\tau)]$ (cf. Theorem 5.4 in [15]). Consequently, when speaking of linear systems, the terms bounded, stable, and finite-gain stable may be used interchangeably.

5. Fundamental Stability Theorem

An abstract result which provides an aggregate characterization of the set feedback system outputs achievable with a specific system input is now stated. We refer to this result as our fundamental stability theorem because stability tests of both the input-output type and the Lyapunov state-space type can be derived from this result. The stability implications of the result are found to have a simple interpretation in terms of 'topological separation' of the product space $X_e \times Y_e$ on which the systems dynamical relations are defined.

Theorem 5.1 (Fundamental Stability Theorem):

Let S be a subset of $X_e \times Y_e$. Suppose that real-valued functionals $d(\underline{x},\ \underline{y},\ \tau)$, $\eta_1(\underline{x},\ \underline{y},\ \tau)$, $\eta_2(u,\ \tau)$, $\eta_3(\underline{v},\ \tau)$ and $\eta_4(\underline{x},\ \underline{y},\ \tau)$ can be found such that for each $\tau \in T$

$$\overline{G}(u) \cap S \subset \{(\underline{x},\ \underline{y}) \mid d(\underline{x},\ \underline{y},\ \tau) \geq \eta_1(\underline{x},\ \underline{y},\ \tau) - \eta_2(\underline{u},\ \tau)\} \tag{5.1}$$

$$\overline{H}(\underline{v}) \cap S^I \subset \{(\underline{y},\ \underline{x}) \mid d(\underline{x},\ \underline{y},\ \tau) \leq \eta_3(\underline{v},\ \tau) - \eta_4(\underline{x},\ \underline{y},\ \tau)\}. \tag{5.2}$$

Then every solution $(\underline{x},\ \underline{y})$ of (2.1) in S satisfies the inequality

$$\eta_{out}(\underline{x},\ \underline{y},\ \tau) \leq \eta_{in}(\underline{u},\ \underline{v},\ \tau) \tag{5.3}$$

for all $\tau \in T$, where

$$\eta_{out} \triangleq \eta_1 + \eta_4 \tag{5.4}$$

$$\eta_{in} \triangleq \eta_2 + \eta_3$$

Proof: Suppose $(\underline{x},\ \underline{y})$ is a solution of (2.1) and that $(\underline{x},\ \underline{y}) \in S$. Then,

$$(\underline{x}, \underline{y}) \in \bar{G}(\underline{u}) \cap \bar{H}^{I}(\underline{v}) \cap S \ . \qquad (5.6)$$

By (5.1) - (5.2) it follows that for all $\tau \in T$

$$d(\underline{x}, \underline{y}, \tau) \geq \eta_1(\underline{x}, \underline{y}, \tau) - \eta_2(\underline{u}, \tau) \qquad (5.7)$$

$$d(\underline{x}, \underline{y}, \tau) \leq \eta_3(\underline{v}, \tau) - \eta_4(\underline{x}, \underline{y}, \tau) \ . \qquad (5.8)$$

Substracting (5.7) from (5.6) and adding η_{in} to both sides yields

$$\begin{aligned} \eta_{in}(\underline{u}, \underline{v}, \tau) \geq\ & (\eta_1(\underline{x}, \underline{y}, \tau) + \eta_4(\underline{x}, \underline{y}, \tau) \\ & - \eta_2(\underline{u}, \tau) - \eta_3(\underline{v}, \tau) \\ & + \eta_{in}(\underline{u}, \underline{v}, \tau)) \\ =\ & \eta_{out}(\underline{x}, \underline{y}, \tau) \end{aligned} \qquad (5.9)$$

which proves Theorem 5.1.

The importance of Theorem 5.1 is that it provides an aggregate characterization of the set of output pairs $(\underline{x}, \underline{y})$ in S that are achievable by input pairs $(\underline{u}, \underline{v})$ in sets of the form $\{(\underline{u}, \underline{v}) \mid \eta_{in}(\underline{u}, \underline{v}, \tau) \leq \text{constant}\}$. By imposing the additional restriction that the functional η_{out} is positive definite on S, this inequality may be used to establish the stability properties of the system (2.1) by establishing a $(\underline{u}, \underline{v})$-dependent bound on $||(\underline{x}, \underline{y})||_\tau$. When, additionally, $S = X_e \times Y_e$ then the stability properties thus determined are <u>global</u>. The following corollary to Theorem 5.1 illustrates this.

<u>Corollary 5.1 (Boundedness & Global Stability)</u>:

a) (Boundedness) If in Theorem 5.1

- i) $S = X_e \times Y_e$,
- ii) η_{out} is positive definite and radially unbounded, and
- iii) η_{in} is bounded,

then the system (2.1) is bounded.

b) (Global Stability) If in Theorem 5.1

- i) $S = X_e \times Y_e$,
- ii) η_{out} is positive definite and radially unbounded, and
- iii) η_{in} is decrescent,

then the system (2.1) is globally stable.

c) (Global Finite-Gain Stability) If in Theorem 5.1

- i) $S = X_e \times Y_e$
- ii) for some strictly positive constants ε_1, ε_2, and α and for all $(\underline{x}, \underline{y}) \in X_e \times Y_e$

$$\left.\begin{aligned} \eta_{out}(\underline{x}, \underline{y}, \tau) &\geq \varepsilon_1(||(\underline{x}, \underline{y})||_\tau)^\alpha \\ \eta_{in}(\underline{u}, \underline{v}, \tau) &\leq \varepsilon_2(||(\underline{u}, \underline{v})||_\tau)^\alpha \ , \end{aligned}\right\} \qquad (5.10)$$

then the system (2.1) is globally finite-gain stable.

<u>Proof</u>: We prove each of the results (a)-(c) in sequence.

a) Since η_{out} is positive-definite radially-unbounded and since η_{in} is bounded, there exist continuous non-decreasing functions $\phi_{out}: R_+ \to R_+$ and $\phi_{in}: R_+ \to R_+$ with $\phi_{out}(0) = (0)$ and ϕ_{out} strictly increasing and $\lim_{\alpha\to\infty} \phi_{out}(\alpha) = \infty$ such that

$$\phi_{out}(||(\underline{x}, \underline{y})||_\tau) \leq \eta_{out}(\underline{x}, \underline{y}, \tau) \qquad (5.11)$$

and

$$\eta_{in}(\underline{u}, \underline{v}, \tau) \leq \phi_{in}(||(\underline{u}, \underline{v})||_\tau) \ . \qquad (5.12)$$

Evidently ϕ_{out}^{-1} exists and is in class K, so

$$||(\underline{x}, \underline{y})||_\tau \leq \phi(||(\underline{u}, \underline{v})||_\tau) \qquad (5.13)$$

where $\phi = \phi_{out}^{-1} \circ \phi_{in}$. Clearly ϕ is continuous and non-decreasing, so the system (2.1) is bounded.

b) Taking ϕ_{out}, ϕ_{in} and ϕ as in (a), it follows that

$$||(x, y)||_\tau \leq \phi(||(\underline{u}, \underline{v})||_\tau) \qquad (5.14)$$

Since η_{out} is positive definite and radially unbounded and since η_{in} is decrescent, it follows that ϕ_{out}^{-1} and ϕ_{in} are decrescent and hence $\phi = \phi_{out}^{-1} \circ \phi_{in}$ is likewise decrescent. It follows from (5.14) that the null solution of (2.1) is globally stable.

c) From (5.10) and the inequality (5.3), it follows that

$$\begin{aligned} \varepsilon_1(||(\underline{x}, \underline{y})||_\tau)^\alpha \leq \eta_{out}(\underline{x}, \underline{y}, \tau) &\leq \eta_{in}(\underline{u}, \underline{v}, \tau) \\ &\leq \varepsilon_2(||(\underline{u}, \underline{v})||_\tau)^\alpha \end{aligned} \qquad (5.15)$$

and hence

$$||(\underline{x}, \underline{y})||_\tau \leq (\varepsilon_2/\varepsilon_1)^{1/\alpha}\ ||(\underline{u}, \underline{v})||_\tau \ . \qquad (5.16)$$

It follows that the null solution of (2.1) is globally finite-gain stable.

<u>Comments</u>: The stability conditions of Corollary 5.1 may be interpreted and conceptually motivated in terms of a 'topological separation'. For simplicity we consider only the case of global stability (part (b) of Corollary 5.1) — a similar interpretation is possible for the other parts of Corollary 5.1. We further assume for simplicity that $\eta_4 \equiv 0$ so that $\eta_{out} = \eta_1$ -- this entails no loss of generality since every case may be reduced to this by substituting $d + \eta_4$ for d. For each $\tau \in T$, the functional $d(\underline{x}, \underline{y}, \tau)$ serves to 'topologically separate' the set $X_e \times Y_e/\{(\underline{0}, \underline{0})\}$ into two disjoint regions, viz. the region where $d(\underline{x}, \underline{y}, \tau) > 0$ and the region where $d(\underline{x}, \underline{y}, \tau) \leq 0$, the set $\{(\underline{x}, \underline{y}) | d(\underline{x}, \underline{y}, \tau) = 0\}$ forming the 'boundary'.[5] Condition 5.1 ensures the undisturbed relation $\bar{H}(\underline{0})$ lies entirely in the

[5]We use the term 'topologically separate' loosely here to describe the partitioning of a set into any two disjoint complementary subsets. Strictly speaking, the mathematical definition of a topological separation demands additionally that the two subsets both be <u>open</u> sets [13].

latter region (where $d(\underline{x}, \underline{y}, \tau) \leq 0$) and that every non-zero point of the undisturbed relation $\mathcal{G}^I(\underline{0})$ lies in the complementary region where $d(\underline{x}, \underline{y}, \tau) \geq \eta_1(\underline{x}, \underline{y}, \tau) > 0$. Consequently, the null solution $(\underline{x}, \underline{y}) = (\underline{0}, \underline{0})$ is the unique solution of the undisturbed system (2.1) — this is prerequisite to global stability. We can visualize $|d(\underline{x}, \underline{y}, \tau)|$ as defining the τ-dependent distance of each point $(\underline{x}, \underline{y}) \in X_e \times Y_e$ from the 'boundary' set $\{(\underline{x}, \underline{y}) | d(\underline{x}, \underline{y}, \tau)\} = 0$, the sign of $d(\underline{x}, \underline{y}, \tau)$ determining on which side of the boundary the point lies. The positive definiteness and radial unboundedness of η_1 ensures for every $(\underline{x}, \underline{y}) \in \mathcal{G}^I(\underline{0})$ that this distance grows unboundedly as $||(\underline{x}, \underline{y})||_\tau$ increases and is bounded below by $\eta_1(\underline{x}, \underline{y}, \tau)$. In this conceptual framework, the quantity $\eta_2(\underline{u}, \tau)$ is simply an upper bound on the 'distance' that $\mathcal{G}^I(\underline{u})$ shifts toward the boundary as a consequence of the disturbance $\underline{u}$. Similarly, the 'distance' of $\mathcal{H}(0)$ from the boundary is non-positive and $\eta_3(\underline{v}, \tau)$ is an upper bound on the distance that $\mathcal{H}(\underline{v})$ shifts toward the boundary as a consequence of $\underline{v}$. Because solutions $(\underline{x}, \underline{y})$ of (1.1) must lie in the set $\mathcal{G}^I(\underline{u}) \cap \mathcal{H}(\underline{v})$, it follows that $(\underline{x}, \underline{y})$ is an element of the bounded set

$$\{(\underline{x}, \underline{y}) \in X_e \times Y_e \mid \eta_1(\underline{x}, \underline{y}, \tau) \leq \eta_2(\underline{u}, \tau) + \eta_3(\underline{v}, \tau) \triangleq \eta_{in}(\underline{u}, \underline{v}, \tau)\} \; ; \qquad (5.17)$$

this set is depicted in Figure 6 as the cross-hatched four-pointed 'star' in the center of the figure. The fact that η_{in} and, hence, η_2 and η_3 are decrescent ensures that vanishly small $\underline{u}$ and $\underline{v}$ will produce vanishly small shifts $\phi_1(\underline{u}, \tau)$ and $\phi_2(\underline{v}, \tau)$ in the respective relations $\mathcal{G}^I(\underline{u})$ and $\mathcal{H}(\underline{v})$. This establishes the global stability of the system (5.1).

In view of the foregoing, one may loosely interpret Corollary 5.1 as saying that stability can be assured if one can find some real-valued functional (viz. $d(\underline{x}, \underline{y}, \tau)$) which separates the set $\mathcal{G}^I(\underline{0}) \cup \mathcal{H}(\underline{0})$ less the pair $(\underline{x}, \underline{y}) = (\underline{0}, \underline{0})$ into the component parts $\mathcal{G}^I(\underline{0}) - \{(\underline{0}, \underline{0})\}$ and $\mathcal{H}(\underline{0}) - \{(\underline{0}, \underline{0})\}$. The conditions that η_{in} be decrescent and that η_{out} be positive definite may be viewed as technical conditions that are imposed to rule-out 'peculiar' situations in which either the amount of separation fails to grow with distance from the origin in the $X_e \times Y_e$ 'plane' or in which the system is ill-posed in the sense that small disturbances $(\underline{u}, \underline{v})$ produce disproportionately large changes in the input-output relations $\mathcal{G}(\underline{u})$ and $\mathcal{H}(\underline{v})$.

It is noteworthy that Theorem 5.1 and Corollary 5.1 make no reference to loop transformations, multipliers, contraction mappings, or other mathematical paraphernalia frequently associated with input-output stability results (cf. [3] - [6]). This is in part a consequence of the fact that, in contrast to some previous input-output stability criteria, no fixed-point theorems (e.g., the contraction mapping theorem) are used in the proof of Corollary 5.1 and Theorem 5.1. This underscores the fact that existence of solutions — and existence is always assured when fixed-point theorems are employed — is not central to the issue of stability. Rather, in stability analysis, we are concerned primarily with ascertaining that all existing solutions are stable. Existence of solutions, which relates to the 'well-posedness' of the system equations, can be deduced from entirely separate considerations [4, pp. 93-101].[6]

[6] Well-posedness tests, based on considerations other than stability, are provided in [4]. However, it should be noted that (in contrast to the view taken here and elsewhere in the literature, cf. [1], [6], [14]), reference [4] defines well-posedness to be prerequisite to any discussion of stability or instability.

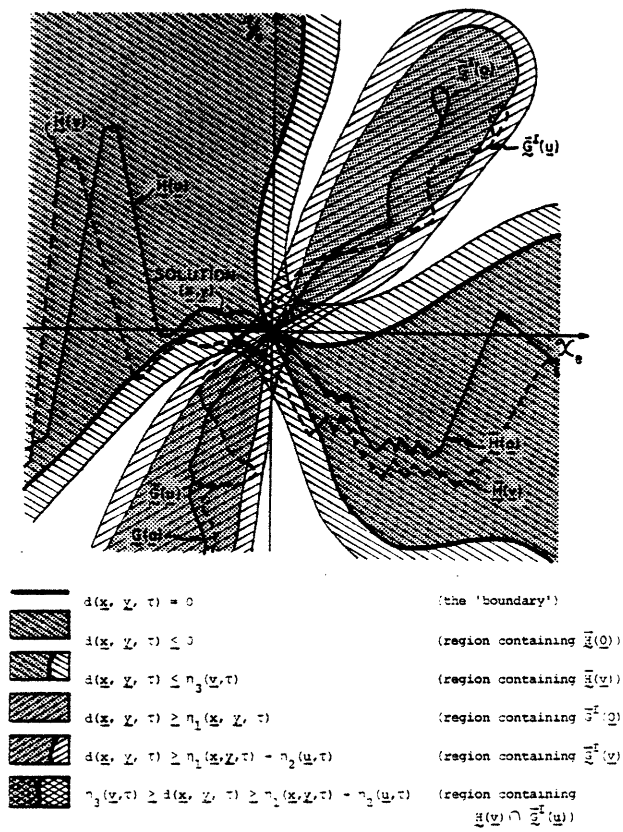


Figure 6. Visualization of the conditions of Theorem 5.1 in the "$X_e \times Y_e$-plane" — the solution $(\underline{x}, \underline{y})$ must lie in the region containing $\mathcal{H}(\underline{v}) \cap \mathcal{G}^I(\underline{u})$.

The Lyapunov stability result discussed in §3 is a special case of Corollary 5.1 in which the separating functional $d(\underline{x}, \underline{y}, \tau)$ is taken to be

$$d(\underline{x}, \underline{y}, \tau) = \int_0^\tau \nabla V(\underline{x}(t), t)\underline{y}(t) + \partial V/\partial t(\underline{x}(t), t)dt$$

where $V(\underline{x}, \tau)$ is the 'Lyapunov' function. With this choice of $d(\underline{x}, \underline{y}, \tau)$ and with $\tilde{G}(\underline{u})$ and $\tilde{H}(\underline{v})$ as in (3.7), it follows

i) $$d(\underline{x}, \underline{y}, \tau) = \int_0^\tau \nabla V(\underline{x}(t), t)\dot{\underline{x}}(t) + \partial V/\partial t(\underline{x}(t), t)dt = V(\underline{x}(\tau), \tau) - V(\underline{x}(0), 0)$$

for all $(\underline{x}, \underline{y})$ satisfying

$$\underline{x} = \tilde{G}(\underline{x}_0) \cdot \underline{y} = \int_0^\tau \underline{y}(t)dt + \underline{x}_0$$

ii) $$d(\underline{x}, \underline{y}, \tau) = \int_0^\tau \nabla V(\underline{x}(t), t)f(\underline{x}(t), t) + \partial V/\partial t(\underline{x}(t), t)dt$$

for all $(\underline{x}, \underline{y})$ satisfying

$$\underline{y}(t) = f(\underline{x}(t), t).$$

Evidently when V is positive definite and decrescent and when $\nabla V(\underline{x}, t)f(\underline{x}, t) + \partial V/\partial t(\underline{x}, t) \leq 0$ for all $\underline{x}$ and t, then the conditions of Corollary 5.1 are satisfied with

$$\eta_1(\underline{x}, \tau) = V(\underline{x}(\tau), \tau)$$

and with

$$\eta_2(\underline{x}_0) = V(\underline{x}_0, 0)$$

$$\eta_3 = \eta_4 \equiv 0 .$$

This establishes that Lyapunov stability results can be treated as corollaries to Theorem 5.1.

Not surprisingly, Zames' powerful conic relation stability theorem can also be shown to be a corollary to Theorem 5.1. To prove this and to demonstrate the power of our results, a generalization of Zames' conic relation theorem is now developed.

6. The Sector Stability Criterion

The conic relation stability theorem of Zames is generalized in this section to permit the utilization of the more flexible definition of sector which follows.

Definition (Sector):

Let X_e and Y_e be extended normed spaces and let Z_e be an extended inner product space. For each $\tau \in T$ let

$$\tilde{F}(\underline{x}, \underline{y}, \tau) \triangleq \langle \tilde{F}_{11}\underline{y} + \tilde{F}_{12}\underline{x}, \tilde{F}_{21}\underline{y} + \tilde{F}_{22}\underline{x} \rangle_\tau \quad (6.1)$$

where $\tilde{F}_{ij}\underline{0} = \underline{0}$ (i, j = 1, 2) and $\tilde{F}_{11}, \tilde{F}_{21}: Y_e \to Z_e$ and $\tilde{F}_{12}, \tilde{F}_{22}: X_e \to Y_e$. Then the sector of $\tilde{F}$ is defined to be

$$\text{Sector } (\tilde{F}) \triangleq \{(\underline{x}, \underline{y}) \in X_e \times Y_e \mid \text{for all } \tau \in T \;\; \tilde{F}(\underline{x}, \underline{y}, \tau) \leq 0\} . \quad (6.2)$$

For notational convenience, the functional $\tilde{F}$ will be denoted equivalently by the 2×2 array

$$\tilde{F} \triangleq \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix} \quad (6.3)$$

Definition (inside, outside, strictly inside, strictly outside):

A subset $\tilde{A}$ of $X_e \times Y_e$ is said to be inside Sector $(\tilde{F})$ if $\tilde{A} \subset \text{Sector } (\tilde{F}) \triangleq \{(\underline{x}, \underline{y}) \in X_e \times Y_e|$ for all $\tau \in T$ $\tilde{F}(\underline{x}, \underline{y}, \tau) \leq 0\}$; $\tilde{A}$ is said to be strictly inside Sector $(\tilde{F})$ if for some $\varepsilon > 0$, $\tilde{A} \subset \{(\underline{x}, \underline{y}) \in X_e \times Y_e|$for all $\tau \in T$ $\tilde{F}(\underline{x}, \underline{y}, \tau) \leq -\varepsilon||(\underline{x}, \underline{y})||_\tau^2\}$; $\tilde{A}$ is said to be outside Sector $(\tilde{F})$ if $\tilde{A} \subset \{(\underline{x}, \underline{y}) \in X_e \times Y_e|$for all $\tau \in T$ $\tilde{F}(\underline{x}, \underline{y}, \tau) \geq 0\}$; and, $\tilde{A}$ is said to be strictly outside Sector $(\tilde{F})$ if for some $\varepsilon > 0$, $\tilde{A} \subset \{(\underline{x}, \underline{y}) \in X_e \times Y_e|$ for all $\tau \in T$, $\tilde{F}(\underline{x}, \underline{y}, \tau) \geq \varepsilon||(\underline{x}, \underline{y})||_\tau^2\}$.

We now state a stability result based on Corollary 5.1 that employes sectors to accomplish the requisite 'topological separation' of the space $X_e \times Y_e$. The proof, which involves a straightforward — but tedious — verification of the conditions of Corollary 5.1, is in Appendix A.

Theorem 6.1 (Sector Stability Criterion):

Let F be a 2×2 array as in (6.3); let the $\tilde{F}_{ij}$(i, j = 1, 2) have finite incremental gain; let the mappings of $\underline{u} \in U$ into $\overline{G}(\underline{u})$ and $\underline{v} \in V$ into $\overline{H}(\underline{v})$ be bounded (respectively, globally stable; respectively globally finite-gain stable) about the respective sets $\overline{G}(\underline{0})$ and $\overline{H}(\underline{0})$. If $\overline{G}^I(\underline{0})$ is strictly inside Sector $(\tilde{F})$ and if $\overline{H}(\underline{0})$ is outside Sector $(\tilde{F})$, then system (1.1) is bounded (respectively, globally stable; respectively, globally finite-gain stable).

Proof: See Appendix A.

Comments: The requirement in Theorem 6.1 that the functions mapping of $\underline{u} \in U$ into $\overline{G}(\underline{u})$ and $\underline{v} \in V$ into $\overline{H}(\underline{v})$ be bounded about the sets $\overline{G}(\underline{0})$ and $\overline{H}(\underline{0})$ should not be confused with the more restrictive requirement that the subsystems $\overline{G}$ and $\overline{H}$ be 'open-loop bounded', i.e., that the mappings of $(\underline{u}, \underline{y})$ into $\overline{G}(\underline{u})\underline{y}$ and $(\underline{v}, \underline{x})$ into $\overline{H}(\underline{v})\underline{x}$ be bounded about $\{(0, 0)\}$. For example, if the disturbances $\underline{u}$ and $\underline{v}$ enter additively as in Figure 3 — and this is the only case considered in the majority of the input-output stability literature — then the boundedness requirement placed on the mappings $\overline{G}(\cdot)$ and $\overline{H}(\cdot)$ in Theorem 6.1 is automatically satisfied (with finite gain!). Thus, the boundedness restriction on the mappings $\overline{G}(\cdot)$ and $\overline{H}(\cdot)$ is actually very mild; it can be viewed as a sort of well-posedness condition on the feedback equations, ensuring that small disturbances do not produce unboundedly large dislocations of the dynamical relations in the $X_e \times Y_e$-'plane' — cf. [4, p. 90] condition WP.4.

Comparison of the definition of Sector $(\tilde{F})$ with Zames' conic sector (3.1) shows that

$$\text{Cone } (c, r) = \text{Sector } \left(\begin{bmatrix} 1 & -(c + r) \\ 1 & -(c - r) \end{bmatrix}\right) .$$

Zames' conic relation stability criterion is a special case of Theorem 6.1 that results when sectors of this form are employed and the class of systems considered is restricted to the additive-input type depicted in Figure 3.

Some Properties of Sectors:

Zames' [1, App. A] demonstrates that his conic sectors have several properties which make them especially well-suited to feedback system stability analysis. Our more general sectors have similar properties, some of which are enumerated in the following lemma.

Lemma 6.2 (Sector Properties):

Let $\underset{\sim}{F}_{ij}$ and $\underset{\sim}{F}_{ij}^{(k)}$ be operators mapping into extended inner product spaces Z_e and $Z_e^{(k)}$ respectively; let $\underset{\sim}{F}_{ij}\underline{0} = \underline{0}$ and $\underset{\sim}{F}_{ij}^{(k)}\underline{0} = \underline{0}$; let $\overline{\underset{\sim}{A}}$, $\underset{\sim}{B}$ and $\overline{\underset{\sim}{A}}^{(k)}$ be relations on extended normed spaces; let $(\underline{0}, \underline{0}) \in \overline{\underset{\sim}{A}}, \overline{\underset{\sim}{B}}, \overline{\underset{\sim}{A}}^{(k)}$; let a and b be scalers with ab > 0; let $\underset{\sim}{M}$ and $\underset{\sim}{M}^*$ be operators with the property that $\langle \underset{\sim}{M}\underline{z}_1, \underline{z}_2\rangle_\tau = \langle \underline{z}_1, \underset{\sim}{M}^*\underline{z}_2\rangle_\tau$ for all $\underline{z}_1$, $\underline{z}_2 \in Z_e$ and all $\tau \in T$. Then the following properties hold:

i) (Complimentary Sector)

$$\overline{\underset{\sim}{A}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix}\right) \Longleftrightarrow \overline{\underset{\sim}{A}} \ \underline{\text{outside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ -\underset{\sim}{F}_{21} & -\underset{\sim}{F}_{22} \end{bmatrix}\right) \quad (6.4)$$

furthermore, (6.4) holds with inside and outside replaced respectively by strictly inside and strictly outside.

ii) (Multiplier)

$$\text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{M}\cdot\underset{\sim}{F}_{21} & \underset{\sim}{M}\cdot\underset{\sim}{F}_{22} \end{bmatrix}\right) = \text{Sector} \left(\begin{bmatrix} a\ \underset{\sim}{M}^*\ \underset{\sim}{F}_{11} & a\ \underset{\sim}{M}^*\ \underset{\sim}{F}_{12} \\ b\ \underset{\sim}{F}_{21} & b\ \underset{\sim}{F}_{22} \end{bmatrix}\right) \quad (6.5)$$

iii) (Inverse Relation)

$$\overline{\underset{\sim}{A}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix}\right) \Longleftrightarrow \overline{\underset{\sim}{A}}^{I} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{12} & \underset{\sim}{F}_{11} \\ \underset{\sim}{F}_{22} & \underset{\sim}{F}_{21} \end{bmatrix}\right) \quad (6.6)$$

Furthermore, (6.6) holds with inside replaced by strictly inside throughout.

iv) (Sums of Relations) If $\overline{\underset{\sim}{B}}$ = Graph ($\underset{\sim}{B}$) and if $\underset{\sim}{F}_{11}$ and $\underset{\sim}{F}_{21}$ are linear, then

$$\overline{\underset{\sim}{A}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix}\right) \Longrightarrow (\overline{\underset{\sim}{A}} - \overline{\underset{\sim}{B}}) \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{11}\cdot\underset{\sim}{B} + \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{21}\cdot\underset{\sim}{B} + \underset{\sim}{F}_{22} \end{bmatrix}\right) \quad (6.7)$$

If $(\overline{\underset{\sim}{A}} - \overline{\underset{\sim}{B}})$ has finite gain, then (6.7) holds with inside replaced by strictly inside throughout.

v) (Composition Products of Relations)

a) If $\overline{\underset{\sim}{B}}$ = Graph ($\underset{\sim}{B}$), then

$$\overline{\underset{\sim}{B}} \circ \overline{\underset{\sim}{A}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix}\right) \Longrightarrow \overline{\underset{\sim}{A}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11}\cdot\underset{\sim}{B} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21}\cdot\underset{\sim}{B} & \underset{\sim}{F}_{22} \end{bmatrix}\right) \quad (6.8)$$

b) If $\overline{\underset{\sim}{A}}$ = Graph ($\underset{\sim}{A}$) and if $\underset{\sim}{A}^{-1}$ exists, then

$$\overline{\underset{\sim}{B}} \circ \overline{\underset{\sim}{A}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix}\right) \Longrightarrow \overline{\underset{\sim}{B}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12}\cdot\underset{\sim}{A}^{-1} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22}\cdot\underset{\sim}{A}^{-1} \end{bmatrix}\right) \quad (6.9)$$

c) If $\overline{\underset{\sim}{A}}$ = Graph ($\underset{\sim}{A}$), then

$$\overline{\underset{\sim}{B}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11}\cdot\underset{\sim}{A} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21}\cdot\underset{\sim}{A} & \underset{\sim}{F}_{22} \end{bmatrix}\right) \Longrightarrow \overline{\underset{\sim}{A}} \circ \overline{\underset{\sim}{B}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix}\right) \quad (6.10)$$

Furthermore, if $\overline{\underset{\sim}{A}}$ has finite gain, then (6.8) - (6.10) hold with inside replaced by strictly inside throughout.

vi) (Composites of Relations)

Suppose $\overline{\underset{\sim}{A}} = \{((\underline{x}^{(1)}, \ldots, \underline{x}^{(n)}), (\underline{y}^{(1)}, \ldots, \underline{y}^{(n)}) \mid (\underline{x}^{(k)}, \underline{y}^{(k)}) \in \overline{\underset{\sim}{A}}^{(k)}$ for all $k = 1, \ldots, n\}$; suppose $\underset{\sim}{F}_{ij}(\underline{\xi}^{(1)}, \ldots, \underline{\xi}^{(n)}) \triangleq (\underset{\sim}{F}_{ij}^{(1)}\underline{\xi}^{(1)}, \ldots, \underset{\sim}{F}_{ij}^{(n)}\underline{\xi}^{(n)})$ for (i, j = 1, 2); and suppose that $Z_e = Z_e^{(1)} \times \ldots \times Z_e^{(n)}$ and that the associated inner-products satisfy

$$\langle(\underline{z}_1^{(1)}, \ldots, \underline{z}_1^{(n)}), (\underline{z}_2^{(1)}, \ldots, \underline{z}_2^{(n)})\rangle_\tau = \sum_{k=1}^{n} \langle\underline{z}_1^{(k)}, \underline{z}_2^{(k)}\rangle_\tau .$$

Then,

$$\overline{\underset{\sim}{A}} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11} & \underset{\sim}{F}_{12} \\ \underset{\sim}{F}_{21} & \underset{\sim}{F}_{22} \end{bmatrix}\right) \Longleftrightarrow \overline{\underset{\sim}{A}}^{(k)} \ \underline{\text{inside}} \ \text{Sector} \left(\begin{bmatrix} \underset{\sim}{F}_{11}^{(k)} & \underset{\sim}{F}_{12}^{(k)} \\ \underset{\sim}{F}_{21}^{(k)} & \underset{\sim}{F}_{21}^{(k)} \end{bmatrix}\right)$$

for all k = 1, ..., n. (6.11)

Furthermore, (6.11) holds if inside is replaced by strictly inside throughout.

vii) Properties (i) - (vi) hold if throughout inside is replaced by outside and strictly inside is replaced by strictly outside.

Proof: See Appendix B.

Comment: Properties (i) and (ii) of Lemma (6.2) provide a parameterization of the various representations of a sector. Property (iii) establishes the relation between sectors containing $\underset{\sim}{A}$ and sectors containing the inverse relation $\underset{\sim}{A}^I$. Properties (iv) - (vi) permit the characterization of a sector containing the relation of a complicated multivariable system using knowledge of sectors associated with subsystems comprising the system and knowledge of the interconnection structure of the system — these properties have application to the analysis of the stability of multivariable systems.

7. Discussion

Our fundamental stability theorem (Theorem 5.1) is of significance largely because it permits a clear understanding of the basic issues involved in stability analysis; this is enabled by the relatively high level of abstraction in our formulation of the stability problem. In particular, Corollary 5.1 makes it clear that if one can find a partitioning of the product space $X_e \times Y_e$ such that the undisturbed relation $\underset{\sim}{G}^I(\underline{0})$ lies in one part of the separated space and the undisturbed relation $\underset{\sim}{H}(\underline{0})$ lies 'strictly' in the other part, then under mild conditions closed-loop stability can be concluded. The conceptual simplicity of this abstract view of stability theory may prove to be of considerable pedagogical value, since it is possible to relate the conditions of the various input-output and Lyapunov stability criteria to the simple geometric notion of topological separation. Also, the conceptual simplicity of the topological separation viewpoint serves to elucidate the general approach that must be taken to generate new results such as the sector stability criterion (Theorem 6.1). Our sector stability criterion leads fairly directly to powerful new multivariable generalizations of the circle stability criterion for nonlinear systems; these generalizations are developed in [9] and are the subject of a forthcoming paper.

One of the implications of using sets (such as the components of a topological separation) to bound the dynamical relations of a system is the possibility of determining the qualitative behavior of a system — and even bounding its quantitative behavior — using only a coarse knowledge of the system. The stability margins (e.g., gain margin and phase margin) of a simplified model of a feedback system can be thusly characterized, providing a measure of robustness against the destabilizing effects of modeling errors — this is demonstrated in [9]. Also, bounds on a poorly defined or imprecisely modelled system's transient response can be found using the inequality (5.3), allowing one to deduce the relative degree of exponential stability or instability.

Appendix A: Proof of 6.1

We apply Corollary 5.1, taking $d(\underline{x}, \underline{y}, \tau)$ to be the map

$$d(\underline{x}, \underline{y}, \tau) = \begin{cases} 0 & , \text{ if } \|(\underline{x},\underline{y})\|_\tau = 0 \\ (1/\|(\underline{x}, \underline{y})\|_\tau) F(\underline{x}, \underline{y}, \tau), & \text{if } \|(\underline{x},\underline{y})\|_\tau \neq 0 \end{cases} \quad \text{(A1)}$$

We begin by establishing (5.1). Let $\underline{u} \in U$ be fixed and take $(\underline{x}, \underline{y}) \in \underset{\sim}{G}^I(\underline{u})$. Applying the Schwartz inequality, we have that for some $\varepsilon > 0$, for every $(\underline{x}_0, \underline{y}_0)$ $\underset{\sim}{G}^I(\underline{0})$, and every $\tau \in T$

$$\begin{aligned} \|(\underline{x}, \underline{y})\|_\tau \cdot d(\underline{x}, \underline{y}, \tau) &= -\langle(\underset{\sim}{F}_{11}\underline{y} + \underset{\sim}{F}_{12}\underline{x}), \\ &\quad (\underset{\sim}{F}_{21}\underline{y} + \underset{\sim}{F}_{22}\underline{x}\rangle_\tau \\ &= \|(\underline{x}_0, \underline{y}_0)\|_\tau \cdot d(\underline{x}_0, \underline{y}_0, \tau) \\ &\quad - \langle(\underset{\sim}{F}_{11}\underline{y}_0 + \underset{\sim}{F}_{12}\underline{x}_0), [(\underset{\sim}{F}_{21}\underline{y} - \underset{\sim}{F}_{21}\underline{y}_0) \\ &\quad + (\underset{\sim}{F}_{22}\underline{x} - \underset{\sim}{F}_{22}\underline{x}_0)]\rangle_\tau \\ &\quad - \langle[(\underset{\sim}{F}_{11}\underline{y} - \underset{\sim}{F}_{11}\underline{y}_0) + \underset{\sim}{F}_{12}\underline{x} - \underset{\sim}{F}_{12}\underline{x}_0)], \\ &\quad (\underset{\sim}{F}_{21}\underline{y} - \underset{\sim}{F}_{22}\underline{x})\rangle_\tau \\ &\geq \varepsilon\|(\underline{x}_0, \underline{y}_0)\|_\tau^2 \\ &\quad - \|(\underset{\sim}{F}_{11}\underline{y}_0 + \underset{\sim}{F}_{12}\underline{x}_0)\|_\tau \cdot \| \\ &\quad [\underset{\sim}{F}_{21}\underline{y} - \underset{\sim}{F}_{21}\underline{y}_0) + (\underset{\sim}{F}_{22}\underline{x} - \underset{\sim}{F}_{22}\underline{x}_0)]\|_\tau \\ &\quad - \|[(\underset{\sim}{F}_{11}\underline{y} - \underset{\sim}{F}_{11}\underline{y}_0) + (\underset{\sim}{F}_{12}\underline{x} - \\ &\quad \underset{\sim}{F}_{12}\underline{x}_0)]\|_\tau \cdot \|(\underset{\sim}{F}_{21}\underline{y} - \underset{\sim}{F}_{22}\underline{x})\|_\tau \\ &\geq \varepsilon\|(\underline{x}_0, \underline{y}_0)\|_\tau^2 \\ &\quad - (k\|(\underline{x}_0, \underline{y}_0)\|_\tau) \cdot (k\|[(\underline{x}, \underline{y}) - \\ &\quad (\underline{x}_0, \underline{y}_0)]\|_\tau) \\ &\quad - (k\|[(\underline{x}, \underline{y}) - (\underline{x}_0, \underline{y}_0)\|_\tau) \cdot \\ &\quad (k\|(\underline{x}, \underline{y})\|_\tau) \end{aligned} \quad \text{(A2)}$$

where the latter inequality follows with $k < \infty$ an upper bound on the gain and incremental gain of $\underset{\sim}{F}_{ij}$ (i, j = 1, 2). Since by hypothesis the map of $\underline{u} \in U$ into $\underset{\sim}{G}(u)$ is bounded, there exist a continuous increasing function $\rho_1: R_+ \to R_+$ and a point $(\underline{x}_0^{(1)}, \underline{y}_0^{(1)}) \in \underset{\sim}{G}^I(\underline{0})$ such that for every $\tau \in T$

$$\|[(\underline{x}, \underline{y}) - (\underline{x}_0^{(1)}, \underline{y}_0^{(1)})]\|_\tau \leq \rho_1(\|\underline{u}\|_\tau). \quad \text{(A3a)}$$

Also, for all $\tau \in T$, there exists a point $(\underline{x}_0^{(2)}, \underline{y}_0^{(2)}) \in \underset{\sim}{G}^I(\underline{0})$ such that

$$\|[(\underline{x}, \underline{y}) - (\underline{x}_0^{(2)}, \underline{y}_0^{(2)})]\|_\tau \leq \|(\underline{x}, \underline{y})\|_\tau, \quad \text{(A3b)}$$

namely the point $(\underline{x}_0^{(2)}, \underline{y}_0^{(2)}) = (\underline{0}, \underline{0})$. From (A3a, b) it follows that there exists an $(\underline{x}_0, \underline{y}_0) \in \underset{\sim}{G}^I(\underline{0})$ such that for all $\tau \in T$

$$\|[(\underline{x}, \underline{y}) - (\underline{x}_0, \underline{y}_0)]\|_\tau \leq \min\{\|(\underline{x}, \underline{y})\|_\tau, \rho_1(\|\underline{u}\|_\tau\}, \quad \text{(A4)}$$

$$\|(\underline{x}, \underline{y})\|_\tau - \rho_1(\|\underline{u}\|_\tau) \leq \|(\underline{x}_0, \underline{y}_0)\|_\tau \leq 2 \cdot \|(\underline{x}, \underline{y})\|_\tau \quad \text{(A5)}$$

and, from the former inequality in (A5), it follows that for all $\tau \in T$

$$\|(\underline{x}_0, \underline{y}_0)\|_\tau^2 \geq \|\underline{x}, \underline{y})\|_\tau(-2\rho_1(\|\underline{u}\|_\tau) + \|(\underline{x}, \underline{y})\|_\tau). \quad \text{(A6)}$$

Substituting (A4), (A5), and (A6) into (A2) and dividing by $\|(\underline{x}, \underline{y})\|_\tau$, it follows that for all $(\underline{x}, \underline{y}) \in \underset{\sim}{G}^I(\underline{u})$ and all $\tau \in T$,

$$d(\underline{x}, \underline{y}, \tau) \geq \varepsilon(||(\underline{x}, \underline{y})||_\tau - 2\rho_1(||\underline{u}||_\tau)) - 2k^2\rho_1(||\underline{u}||_\tau) - k^2\rho_1(||\underline{u}||_\tau) \quad \text{(A7)}$$

Taking $\phi_1, \phi_2 : R_+ \to R_+$, $\eta_1(\underline{x}, \underline{y}, \tau)$ and $\eta_2(\underline{u}, \tau)$ to be

$$\phi_1(\alpha) \triangleq \varepsilon\cdot\alpha \quad \text{(A8)}$$

$$\phi_2(\alpha) \triangleq (2\varepsilon + 3k^2)\rho_1(\alpha) \quad \text{(A9)}$$

$$\eta_1(\underline{x}, \underline{y}, \tau) = \phi_1(||(\underline{x}, \underline{y})||_\tau) \quad \text{(A10)}$$

$$\eta_2(\underline{u}, \tau) = \phi_2(||\underline{u}||_\tau) \quad \text{(A11)}$$

we see that $\phi_1 \in K$ is linear and radially unbounded and that for all $(\underline{x}, \underline{y}) \in \overline{G}^I(\underline{u})$ and all $\tau \in T$

$$d(\underline{x}, \underline{y}, \tau) \geq \eta_1(\underline{x}, \underline{y}, \tau) - \eta_2(\underline{u}, \tau) \quad \text{(A12)}$$

which establishes (5.1).

)

Proceeding in an analogous fashion to establish (5.2), we have that for some $\rho_2 : R_+ \to R_+$, every $(\underline{x}, \underline{y}) \in \overline{H}(\underline{v})$, and every $\tau \in T$

$$d(\underline{x}, \underline{y}, \tau) \leq \eta_3(\underline{v}, \tau) - \eta_4(\underline{x}, \underline{y}, \tau) \quad \text{(A13)}$$

where $\eta_3(\underline{v},\tau) = \phi_3(||\underline{v}||_\tau) = 3k^2\rho_2(||v||_\tau)$, $\eta_4(\underline{x}, \underline{y}, \tau) \equiv 0$; as before $k < \infty$ is an upper bound on the gain and incremental gain of $F_{ij}(i, j = 1, 2)$ and ρ_2, like ρ_1, is continuous and increasing.

Thus,

$$\eta_{out}(\underline{x}, \underline{y}, \tau) = \varepsilon||\underline{x}, \underline{y}||_\tau \quad \text{(A14)}$$

and

$$\eta_{in}(\underline{u}, \underline{v}, \tau) = (2\varepsilon + 3k^2)\rho_1(||\underline{u}||_\tau) + 3k^2\rho_2(||\underline{v}||_\tau) \quad \text{(A15)}$$

Clearly η_{out} is positive definite, radially unbounded and satisfies the constraint imposed by (5.10) with $\alpha = 1$. Since η_{in} is clearly bounded, it follows from Corollary 5.1-a that (2.1) is bounded. If additionally the maps taking $\underline{u}$ into $\overline{G}(\underline{u})$ and $\underline{v}$ into $\overline{H}$ are globally stable (globally finite-gain stable) about the respective sets $\overline{G}(\underline{0})$ and $\overline{H}(\underline{0})$, then ρ_1 and ρ_2 may be taken to be in class K (may be taken to linear) from which it follows that η_{in} is decrescent (satisfied (5.10) with $\alpha = 1$) and global stability (global finite-gain stability) of (2.1) follows from Corollary 5.1-b (Corollary 5.1-c).

Appendix B: Proof of Lemma 6.2

We prove properties (i) - (vii) in sequence.

<u>Proof of Property (i)</u>:

$$\overline{A} \ \underline{\text{inside}} \ (\underline{\text{strictly inside}}) \ \text{Sector} \left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right) \quad \text{(B1)}$$

$\Longleftrightarrow$

$$\langle F_{11}\underline{y} + F_{12}\underline{x},\ F_{21}\underline{y} + F_{22}\underline{x}\rangle_\tau \leq -\varepsilon||(\underline{x}, \underline{y})||^2_\tau$$

for some $\varepsilon \geq 0$ ($\varepsilon > 0$), all $\tau \in T$, and all $(\underline{x}, \underline{y}) \in \overline{A}$ (B2)

$\Longleftrightarrow$

$$\{\langle F_{11}\underline{y} + F_{12}\underline{x},\ -F_{21}\underline{y} - F_{22}\underline{x}\rangle_\tau \geq \varepsilon||(\underline{x}, \underline{y})||^2_\tau$$

for some $\varepsilon \geq 0$ ($\varepsilon > 0$), all $\tau \in T$ and all $(\underline{x}, \underline{y}) \in \overline{A}$ (B3)

$\Longleftrightarrow$

$$\overline{A} \ \underline{\text{outside}} \ (\underline{\text{strictly outside}}) \ \text{Sector} \left(\begin{vmatrix} F_{11} & F_{12} \\ -F_{21} & -F_{22} \end{vmatrix}\right) \quad \text{(B4)}$$

This proves property (i).

<u>Proof of Property (ii)</u>:

$$(\underline{x}, \underline{y}) \ \text{Sector} \left(\begin{bmatrix} F_{11} & F_{12} \\ M\cdot F_{21} & M\cdot F_{22} \end{bmatrix}\right) \quad \text{(B5)}$$

$\Longleftrightarrow$

$$\langle F_{11}\underline{y} + F_{12}\underline{x},\ M\cdot F_{21}\underline{y} + M\cdot F_{22}\underline{x}\rangle_\tau \leq 0$$

for all $\tau \in T$ (B6)

$\Longleftrightarrow$

$$\langle F_{11}\underline{y}, M\cdot F_{21}\underline{y}\rangle_\tau + \langle F_{11}\underline{y}, M\cdot F_{22}\underline{x}\rangle_\tau + \langle F_{12}\underline{x}, M\cdot F_{21}\underline{y}\rangle_\tau + \langle F_{12}\underline{x}, M\cdot F_{22}\underline{x}\rangle_\tau \leq 0 \ \text{for all } \tau \in T \quad \text{(B7)}$$

$\Longleftrightarrow$

$$\{(1/ab \langle aM^*\cdot F_{11}\underline{y}, bF_{21}\underline{y}\rangle_\tau + (1/ab) \langle aM^*\cdot F_{11}\underline{y}, bF_{22}\underline{x}\rangle_\tau + (1/ab) \langle aM^*\cdot F_{12}\underline{x}, bF_{21}\underline{y}\rangle_\tau + (1/ab) \langle aM^*\cdot F_{12}\underline{x}, bF_{22}\underline{x}\rangle_\tau\} \leq 0 \ \text{for all } \tau \in T \quad \text{(B8)}$$

$\Longleftrightarrow$

$$\langle aM^*\cdot F_{11}\underline{y} + aM^*\cdot F_{12}\underline{x},\ bF_{21}\underline{y} + bF_{22}\underline{x}\rangle_\tau \leq 0 \ \text{for all } \tau \in T \quad \text{(B9)}$$

$\Longleftrightarrow$

$$(\underline{x}, \underline{y}) \in \text{Sector} \left(\begin{bmatrix} a\ M^*\ F_{11} & + \quad a\ M^*\ F_{12} \\ bF & bF \end{bmatrix}\right) \quad \text{(B10)}$$

From (B5) - (B10), property (ii) follows.

<u>Proof of Property (iii)</u>:

$$\overline{A} \ \underline{\text{inside}} \ (\underline{\text{strictly inside}}) \ \text{Sector} \left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right) \quad \text{(B11)}$$

$\Longleftrightarrow$

$$\langle F_{11}\underline{y} + F_{12}\underline{x},\ F_{21}\underline{y} + F_{22}\underline{x}\rangle_\tau \leq -\varepsilon||(\underline{x}, \underline{y})||^2_\tau$$

for some $\varepsilon \geq 0$ ($\varepsilon > 0$), all $\tau \in T$, and all $(\underline{x},\underline{y}) \in \overline{A}$ (B12)

$\Longleftrightarrow$

$$\langle F_{12}\underline{x} + F_{11}\underline{y},\ F_{22}\underline{x} + F_{21}\underline{y}\rangle_\tau \leq -\varepsilon||(\underline{y}, \underline{x})||^2_\tau$$

for some $\varepsilon \geq 0$ ($\varepsilon > 0$), all $\tau \in T$, and all $(\underline{y},\underline{x}) \in A$ (B13)

$$\Longleftrightarrow$$

$\overline{A}^{I}$ <u>inside</u> (<u>strictly inside</u>) Sector $\left(\begin{bmatrix} F_{12} & F_{11} \\ F_{22} & F_{21} \end{bmatrix}\right)$ (B14)

This proves property (iii).

<u>Proof of Property (iv)</u>:

Let $k = \text{Gain}\ (\overline{A} - \overline{B})$. Then,

$\overline{A}$ <u>inside</u> (<u>strictly inside</u>) Sector $\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right)$ (B15)

$$\Longleftrightarrow$$

$<F_{11}\underline{y} + F_{12}\underline{x},\ F_{21}\underline{y} + F_{22}\underline{x}>_{\tau} \leq -\varepsilon||(\underline{x},\ \underline{y})||^{2}_{\tau}$
for all $(\underline{x},\ \underline{y}) \in \overline{A}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$
and all $\tau \in T$ (B16)

$$\Longleftrightarrow$$

$<F_{11}\underline{y} - F_{11}\ B\ \underline{x} + F_{11}\ B\ \underline{x} + F_{12}\underline{x},$
$F_{21}\underline{y} - F_{21}\ B\ \underline{x} + F_{21}\ B\ \underline{x} + F_{22}\underline{x}>_{\tau} \leq -\varepsilon||(\underline{x},\ \underline{y})||^{2}_{\tau}$
for all $(\underline{x},\ \underline{y}) \in \overline{A}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$
and all $\tau \in T$ (B17)

$$\Longleftrightarrow$$

$<F_{11}(\underline{y} - B\ \underline{x}) + (F_{11}\ B + F_{11})\underline{x},$
$F_{21}(\underline{y} - B\ \underline{x}) + (F_{21}\ B + F_{22})\underline{x}>_{\tau} \leq -\varepsilon||(\underline{x},\ \underline{y})||^{2}_{\tau}$
for all $(\underline{x},\ \underline{y}) \in \overline{A}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$,
and all $\tau \in T$ (B18)

$$\Longleftrightarrow$$

$<F_{11}\tilde{\underline{y}} + (F_{11}\cdot B + F_{12})\underline{x},\ F_{21}\tilde{\underline{y}} + (F_{21}\cdot B + F_{22})x>_{\tau}$
$\leq -\varepsilon||(\underline{x},\ \tilde{\underline{y}} + B\ \underline{x})||^{2}_{\tau} \leq -\varepsilon||\underline{x}||^{2}_{\tau}$
$\leq -\varepsilon/1{+}k^{2}\ ||(\underline{x},\ \tilde{\underline{y}})||^{2}_{\tau}$
for all $(\underline{x},\ \tilde{\underline{y}}) \in (\overline{A} - \overline{B})$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$,
and all $\tau \in T$ (B19)

$$\Longrightarrow$$

$(\overline{A} - \overline{B})$ <u>inside</u> Sector $\left(\begin{bmatrix} F_{11} & F_{11}\cdot B + F_{12} \\ F_{21} & F_{21}\cdot B + F_{22} \end{bmatrix}\right)$

and, provided $k < \infty$ and (B15) holds with the parenthetical <u>strictly inside</u>,

$(\overline{A} - \overline{B})$ <u>strictly inside</u> Sector $\left(\begin{bmatrix} F_{11} & F_{11}\ B + F_{12} \\ F_{21} & F_{21}\ B + F_{22} \end{bmatrix}\right)$ (B20)

This proves property (iv).

<u>Proof of Property</u> (v):

Let $k = \text{Gain}\ (\overline{A})$.

(a) $\overline{B} \circ \overline{A}$ <u>inside</u> (strictly inside) Sector $\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right)$ (B21)

$$\Longleftrightarrow$$

$<F_{11}\underline{y} + F_{12}\underline{x},\ F_{21}\underline{y} + F_{22}\underline{x}>_{\tau} \leq -\varepsilon||(\underline{x},\ \underline{y})||^{2}_{\tau}$
for all $(\underline{x},\ \underline{y}) \in \overline{B} \circ \overline{A}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$,
and all $\tau \in T$ (B22)

$$\Longleftrightarrow$$

$<F_{11}B\tilde{\underline{y}} + F_{12}\underline{x},\ F_{21}\ B\ \tilde{\underline{y}} + F_{22}\underline{x}>_{\tau} \leq -\varepsilon||(x,\ B\ \tilde{\underline{y}})||^{2}$
$\leq -\varepsilon||\underline{x}||^{2}_{\tau} \leq -(\varepsilon/(1+k^{2}))||(\underline{x},\ \tilde{\underline{y}})||^{2}_{\tau}$
for all $(\underline{x},\ \underline{y}) \in \overline{A}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$
and all $\tau \in T$ (B23)

$$\Longleftrightarrow$$

$\overline{A}$ <u>inside</u> Sector $\left(\begin{bmatrix} F_{11}\cdot B & F_{12} \\ F_{21}\cdot B & F_{22} \end{bmatrix}\right)$

and, provided $k < \infty$ and (B21) holds with the parenthetical <u>strictly inside</u>,

$\overline{A}$ <u>strictly inside</u> Sector $\left(\begin{bmatrix} F_{11}\ B & F_{12} \\ F_{21}\ B & F_{22} \end{bmatrix}\right)$ (B24)

(b) $\overline{B} \circ \overline{A}$ <u>inside</u> (<u>strictly inside</u>) Sector $\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right)$ (B25)

$$\Longleftrightarrow$$

$<F_{11}\underline{y} + F_{12}\cdot A^{-1}\cdot A\underline{x},\ F_{21}\underline{y} + F_{22}\cdot A^{-1}\cdot A\underline{x}>_{\tau}$
$\leq -\varepsilon||(\underline{x},\ \underline{y})||^{2}_{\tau}$ for all $(\underline{x},\ \underline{y}) \in \overline{B} \circ \overline{A}$,
some $\varepsilon \geq 0$ $(\varepsilon > 0)$, and all $\tau \in T$ (B26)

$$\Longleftrightarrow$$

$<F_{11}\underline{y} + F_{12}A^{-1}\tilde{\underline{x}},\ F_{21}\underline{y} + F_{22}A^{-1}\tilde{\underline{x}}>_{\tau}$
$\leq -\varepsilon||(A^{-1}\tilde{\underline{x}},\ y)||^{2}_{\tau} \leq -\varepsilon(||\underline{y}||^{2}_{\tau} + ||A^{-1}\tilde{\underline{x}}||^{2}_{\tau})$
$\leq -\varepsilon(||y||^{2}_{\tau} + (1/k^{2})||\tilde{\underline{x}}||^{2}_{\tau})$
$\leq -\varepsilon\cdot\min\{1,\ 1/k^{2}\}\cdot||(\underline{x},\ \tilde{\underline{y}})||^{2}_{\tau}$
for all $(\tilde{\underline{x}},\ \underline{y}) \in \overline{B}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$
and all $\tau \in T$ (B27)

$$\Longrightarrow$$

$\overline{B}$ <u>inside</u> Sector $\left(\begin{bmatrix} F_{11} & F_{12}\cdot A^{-1} \\ F_{21} & F_{22}\cdot A^{-1} \end{bmatrix}\right)$ (B28)

and, provided $k < \infty$ and (B25) holds with the parenthetical <u>strictly inside</u>,

$\overline{B}$ <u>strictly inside</u> Sector $\left(\begin{bmatrix} F_{11} & F_{12}\cdot A^{-1} \\ F_{21} & F_{22}\cdot A^{-1} \end{bmatrix}\right)$ (B29)

(c) $\overline{B}$ <u>inside</u> (<u>strictly inside</u>) Sector $\left(\begin{bmatrix} F_{11}\cdot A & F_{12} \\ F_{21}\cdot A & F_{22} \end{bmatrix}\right)$ (B30)

$$\Longleftrightarrow$$

$<F_{11}\cdot A\underline{y} + F_{12}\underline{x},\ F_{21}\cdot A\underline{y} + F_{22}\underline{x}>_{\tau}$
$\leq -\varepsilon||(\underline{x},\ \underline{y}\ ||^{2}_{\tau}$ for all $(\underline{x},\ \underline{y}) \in \overline{B}$,
some $\varepsilon \geq 0$ $(\varepsilon > 0)$, and all $\tau \in T$ (B31)

$$\Longleftrightarrow$$

$$<F_{11}\tilde{y} + F_{12}\underline{x}, F_{21}\tilde{y} + F_{22}\underline{x}>_\tau \leq -\varepsilon||\underline{x}, \underline{y}||_\tau^2$$
$$= -\varepsilon(||\underline{x}||_\tau^2 + ||\underline{y}||_\tau^2) \leq -\varepsilon(||\underline{x}||_\tau^2$$
$$+ 1/k^2 \cdot ||\tilde{y}||_\tau^2) \leq -\varepsilon \cdot \min\{1, 1/k^2\} \cdot ||(\underline{x}, \tilde{y})||_\tau^2$$

for all $(\underline{x}, \tilde{y}) \in \bar{A} \circ \bar{B}$, some $\varepsilon \geq 0$ $(\varepsilon > 0)$, and all $\tau \in T$ (B32)

$$\Longleftrightarrow$$

$\bar{A} \circ \bar{B}$ inside Sector $\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right)$

and, provided $k < \infty$ and (B30) holds with the parenthetical strictly inside,

$\bar{A} \circ \bar{B}$ strictly inside Sector $\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right)$ (B33)

This proves property (v).

Proof of Property (vi):

$\bar{A}^{(k)}$ inside (strictly inside) Sector $\left(\begin{bmatrix} F_{11}^{(k)} & F_{12}^{(k)} \\ F_{21}^{(k)} & F_{22}^{(k)} \end{bmatrix}\right)$ (B34)

for all $k = 1, \ldots, n$

$$\Longleftrightarrow$$

$<F_{11}^{(k)} \underline{y}^{(k)} + F_{12} \underline{x}^{(k)}, F_{21}^{(k)} \underline{y}^{(k)} + F_{22}^{(k)} \underline{x}>_\tau$ $\leq -\varepsilon^{(k)}||(\underline{x}^{(k)}, \underline{y}^{(k)})||_\tau^2$ for all $(\underline{x}^{(k)}, \underline{y}^{(k)})$ $\bar{A}^{(k)}$, some $\varepsilon^{(k)} \geq 0$ $(\varepsilon^{(k)} > 0)$, all $\tau \in T$, and all $k = 1, \ldots, n$ (B35)

$\sum_{k=1}^{n} <F_{11}^{(k)} \underline{y}^{(k)} + F_{12}^{(k)} \underline{x}^{(k)}, F_{21}^{(k)} \underline{y}^{(k)} + F_{22}^{(k)}>_\tau \leq -\min\{\varepsilon^{(k)} \mid k=1, \ldots, n\}$ $\cdot \sum_{k=1}^{n} ||(\underline{x}^{(k)}, \underline{y}^{(k)}||_\tau^2 = -\min\{\varepsilon^{(k)} \mid k = 1, \ldots, n\} \cdot ||((\underline{x}^{(1)}, \ldots, \underline{x}^{(n)}), (\underline{y}^{(1)}, \ldots, \underline{y}^{(n)})$ for all $((x^{(1)}, \ldots, x^{(n)}), (\underline{y}^{(1)}, \ldots, \underline{y}^{(n)})) \in \bar{A}$, for some $\varepsilon^{(k)} \geq 0$ $(\varepsilon^{(k)} > 0)$ $k = 1, \ldots, n$, and for all $\tau \in T$ (B36)

$$\Longleftrightarrow$$

$\bar{A}$ inside (strictly inside) Sector $\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\right)$ (B37)

This proves property (vi).

Proof of Property (vii):

This follows directly from property (i).

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